RESEARCH STATEMENT

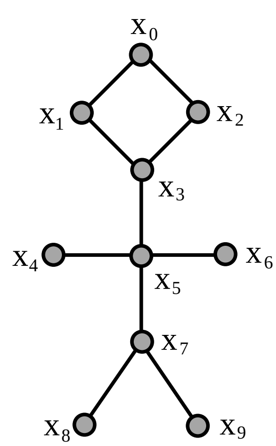
RACHELLE BOUCHAT

My research area is commutative algebra and its interactions with topology, combinatorics, and geometry. Most recently, I have been studying the minimal free resolutions of monomial ideals. Monomial ideals are an important class of ideals because of their interconnections with combinatorics and simplicial topology. Furthermore, monomial ideals occur as Gröbner degenerations of more general ideals generated by polynomials. My approach is to study specific classes of monomial ideals associated to particular classes of graphs. I then look for relationships between the physical structure of the planar graph and the algebraic invariants of the associated ideal related to its minimal free resolution.

1. Edge Ideals

Given an undirected graph G with no loops, we can associate to the planar graph G an ideal I_G. To do this, we first label the vertices of the graph G with variables $x_0, x_1, \ldots, x_n$. Then we consider the ideal I_G generated by x_ix_j where x_i and x_j are the endpoints of an edge in the graph G. This ideal lies in the polynomial ring $S := k[x_0, x_1, \ldots, x_n]$ over the field k and is called the *edge ideal* of G.

Example 1. Consider the graph



Then the corresponding edge ideal is given by

$$I = (x_0x_1, x_0x_2, x_1x_3, x_2x_3, x_3x_5, x_4x_5, x_5x_6, x_5x_7, x_7x_8, x_7x_9)$$

Moreover, $I \subset S := k[x_0, x_1, \ldots, x_9]$.

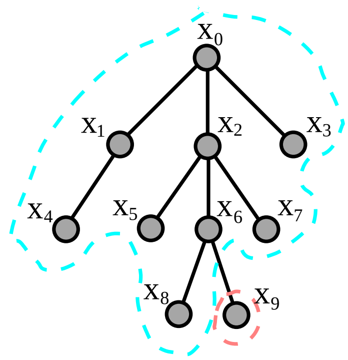
The algebraic invariants of I_G that I have been studying are related to the minimal free resolution of S/I_G. It should be noted that for an arbitrary module M in a ring R, the minimal free resolution of M need not be finite. However, in the case where M is a module of the polynomial ring $S = k[x_0, \ldots, x_n]$ over the field k, Hilbert's syzygy theorem states that the length of the minimal free resolution is at most $n + 1$.

2. Edge Ideals of Trees

Recall that a *tree*, T, is a connected graph with no cycles, and a vertex of degree 1 is called a *leaf* of the tree. By removing a leaf of the tree T, we can decompose the tree into several smaller trees. Using this method of decomposing the trees, one can get the following result regarding the minimal free resolution of the edge ideals corresponding to trees.

Theorem 2. *Given a tree, T, there is a recursive relationship that describes the minimal free resolution of S/I_T in terms of the edge ideals of several smaller subtrees of T.*

Example 3. Consider the tree T below with leaf x_9.



Then the tree T' circled in blue is one subtree that is used in the recursion to determine the minimal free resolution of S/I_T. The other component of the recursion comes from a subforest (a *forest* is a disjoint union of trees) of T related to the ideals $I_{T'}$ and (x_6x_9).

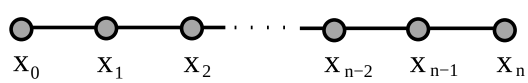
In general, the computation of the minimal free resolution of an ideal is not given by a polynomial time algorithm. However, using the technique described above for decomposing a tree by removing a leaf of the tree we get the following.

Theorem 4. *Given a tree T, there is a polynomial time algorithm for determining the free modules of the minimal free resolution of S/I_T.*

This algorithm can be done rather quickly by hand with the planar graph for small examples, but I have also implemented it in Python for use in Sage (a free open source mathematics' program) enabling it to handle much larger trees.

3. Edge Ideals of Paths

One can specialize even further by considering the monomial ideals given by edge ideals of paths. Recall that an *n-length path*, P_n, looks like



I should note that the minimal free resolution for S/I_{P_n} is not explicitly known. However, using the techniques described above for trees provides some further results concerning I_{P_n}. In the following result we will talk about two different properties. The first property known as *projective dimension* is the length of the minimal free resolution of S/I_{P_n}. The second property is *Castelnuovo-Mumford regularity* which is a measure of how hard it will be to compute a minimal free resolution of S/I_{P_n}, and it also puts a bound on the largest degree of a matrix entry representing a map in the minimal free resolution of S/I_{P_n}.

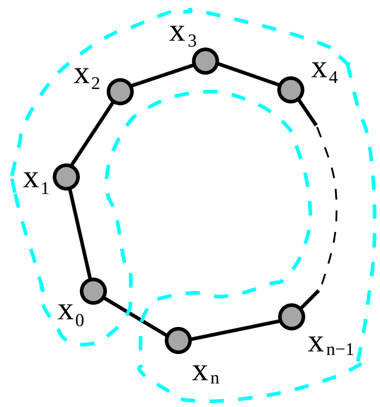
Proposition 5. *Let P_n denote an n-length path. Then the projective dimension of $S/I_{P_n} = \lceil \frac{2n}{3} \rceil$ and the Castelnuovo-Mumford regularity of $S/I_{P_n} = \lceil \frac{n}{3} \rceil$.*

Additionally, using the graph theoretical property of a minimal vertex cover, I get the following result concerning the dimension of S/I_{P_n}.

Proposition 6. *Let P_n denote an n-length path. Then the dimension of $S/I_{P_n} = \lceil \frac{n}{2} \rceil$.*

4. Edge Ideals of Cycles

The techniques that were used to generate the above results for trees and paths were based upon the decomposition of the tree by removing a leaf. For more general graphs, there may not always be a leaf to choose at each stage, because the graph may contain cycles. However, similar techniques can be used to study the minimal free resolution of general undirected graphs. In the case of the general graphs, we will remove any one edge of the graph. Consider the graph of a cycle. After removing an edge from a cycle of length $n+1$, we are left with a path of length n.



Using the result about the projective dimension of an n-length path, we get the following result concerning the edge ideals of cycles.

Proposition 7. *Let C_n be an $(n+1)$-length cycle. Then*

$$\textit{projective dimension of } S/I_{C_n} = \begin{cases} \lceil \frac{2n}{3} \rceil + 1 & \textit{if } 3|n \\ \lceil \frac{2n}{3} \rceil & \textit{else} \end{cases}$$

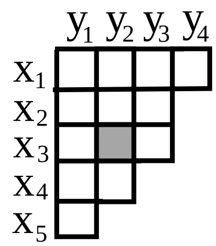
Notice that the ideals I_{C_n} and I_{P_n} only differ by one generator, namely I_{C_n} has the additional generator $x_0 x_n$. In general, by adding an extra generator we would expect the projective dimension to increase. However, we can see by the above result that most of the time the projective dimension is not affected by the addition of this new generator, and in the worst case (namely when $3|n$) the projective dimension only increases by 1.

Since general graphs are composed of paths and cycles, knowing information about paths and cycles provides a way to study the minimal free resolutions associated with the edge ideals of more general graphs.

5. Toric Ideals Associated to Ferrer's Graphs

Currently, I am investigating ideals that correspond to the class of bipartite graphs known as Ferrer's graphs. Ferrer's graphs can be represented by a tableau that takes the form of a matrix with a staircase cut out on the right hand side. Each box of this partial matrix represents an edge of the Ferrer's graph.

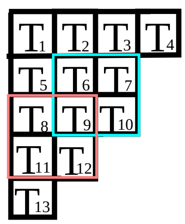
Example 8. The following is a Ferrer's tableau.



The shaded box represents the edge x_3y_2 of the Ferrer's graph.

Considering the tableau as a matrix, one can talk about the ideal generated by the 2-minors of the tableau. This ideal, I, is a toric ideal that defines the special fiber ring of the edge ideal of the Ferrer's graph. To generate this ideal, one must first fill in the entries of the tableau with variables, T_i.

Example 9. After labeling the Ferrer's tableau from above, we get



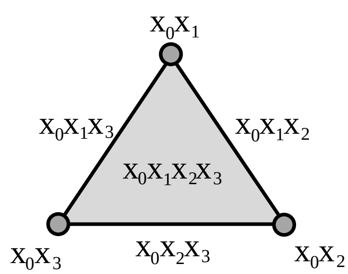
Then we see that $T_6T_{10} - T_7T_9$ and $T_8T_{12} - T_9T_{11}$ are generators of the ideal I.

I am interested in looking at initial ideals of these toric ideals. What this does is simplify the ideal from being toric to being a monomial ideal generated by quadrics. I am then using liaison theory to study these initial ideals and relate the information back to the original toric ideals.

6. Future Research

In the future I would like to extend my work with edge ideals of trees. One way I would like to extend this work is by giving a geometric description for the minimal free resolutions of the edge ideals of paths. The following is an example of such a description.

Example 10. Consider the ideal $I = (x_0x_1, x_0x_2, x_0x_3) \subset S := K[x_0, x_1, x_2, x_3]$. Then the minimal free resolution of S/I can be encoded as



The correspondence between the geometric picture above and the minimal free resolution of S/I is given by the following. The 1^{st} Betti number corresponds to the number of vertices, the 2^{nd} Betti number corresponds to the number of edges, and the 3^{rd} Betti number corresponds to the

2-dimensional face. The projective dimension of S/I is 3 which is the dimension of the highest face of the geometric representation plus 1.

It should be noted that not every monomial ideal has such a geometric description of its minimal free resolution. However, I have further information about the Betti numbers of the edge ideals of paths that suggests a cellular description for the minimal free resolution of paths is likely to exist.

I am also interested in the general study of commutative algebra and am open to looking at other unrelated problems.

7. Incorporating Undergraduates in My Research

Mentoring undergraduate research projects is a part of teaching that I am very excited about. In my undergraduate studies, I remember being assigned a group project in calculus that stepped us through calculating the rainbow angle. The teacher required that each group submit a typed paper complete with appendices and diagrams demonstrating their calculations and knowledge of the topic. I can still remember the excitement that came upon completing the project and feeling like I had really discovered something. It was this early introduction to research in mathematics that started my love of the subject. I would like to incorporate similar guided group projects in the classes I teach to give students an early idea of what research in mathematics entails. My primary motive is to help undergraduate students have their own revelations regarding how and why they lead their mathematical and intellectual lives, regardless of whether they will attend graduate school, take jobs as K-12 teachers, go into industry, or follow any other path.

I firmly believe that students should choose to do research in the area of mathematics that interests them most. My role should be as a guide and a mentor. I am comfortable and confident mentoring undergraduate research in any area of mathematics. I do not need to be an expert in order to help students become experts themselves. However, if a student were to ask me for a research topic, I would first direct them to the study of edge ideals of graphs. There are many questions that are unanswered in this area, and the simple geometric representation of the graphs is inviting for undergraduate investigations.

References

[1] David Bayer and Bernd Sturmfels, *Cellular Resolution of Monomial Modules*, J. Reine Angew. Math. **502** (1998), 123-140.

[2] Alberto Corso and Uwe Nagel, *Monomial and Toric Ideals Associated to Ferrers Graphs*, Trans. Amer. Math. Soc., arXiv:math.AC/0609371 (to appear).

[3] Tai Huy Ha and Adam Van Tuyl, *Splittable Ideals and the Resolutions of Monomial Ideals*, J. Algebra **309**, no. 1, 405-425.

[4] Ezra Miller and Bernd Sturmfels, Combinatorial Commutative Algebra, Springer-Verlag, 2005.

[5] Mauricio Velasco, *Minimal Free Resolutions That Are Not Supported by a CW-Complex*, preprint, 2007.

Department of Mathematics, University of Kentucky, Lexington, KY 40515

E-mail address: `rbouchat@ms.uky.edu`